Formulation of the Global Equations of Motion of a Deformable Body

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The global equations of motion for a deforming body are cast in a new form that is attractive for many studies. The deforming body is assumed a continuum, but its constitutive behavior is left totally general. Equations treated represent the conservation of linear and angular momenta and kinetic energy. The motion (including deformation) of the body is of unrestricted magnitude. A translating and rotating reference frame is defined in terms of the linear and angular momenta of the body, and the global equations of motion are transformed in terms of parameters that are defined relative to this reference frame. It is defined so that the body possesses zero momenta as seen by an observer fixed in the reference frame. It also is shown that, for a given motion, this choice of reference frame minimizes the kinetic energy of the body measured relative to the reference frame. The global equations are represented in a form similar to the classical rigid body equations, which are shown to follow on the assumption of zero stretching. Finally, the conditions for "steady" motion are developed. Solutions are presented for steady motion and for the case when the angular velocity has only steady direction.

Nomenclature

$\bar{c}$	= position vector of origin of noninertial reference frame relative to origin of the inertial frame
C	= constant of integration
da	= differential area of surface
dm	= differential mass of body
dv	= differential volume of body
d/dt	= time derivative relative to the inertial frame
$\bar{\bar{D}}$	= stretching dyadic
$\bar{f}$	= vector body force
$\bar{F}$	= net force applied to the body
$\bar{H}^{[0]}$	= angular momentum of the body measured about the origin of the inertial frame
$\bar{\bar{I}}$	= instantaneous inertia dyadic of the body measured relative to the origin of the noninertial frame
K	= kinetic energy of the body
$\bar{L}^{[0]}$	= net torque applied to the body measured about the origin of the inertial frame
M	= mass of the body
$\bar{n}$	= arbitrary unit vector
$\bar{n}_\omega$	= unit vector in direction of ω
0	= origin of the inertial frame
$\bar{p}$	= position vector measured in the inertial frame
$\bar{P}$	= linear momentum of the body
S	= surface area of the body which follows the body as it deforms
$\bar{\bar{T}}$	= stress dyadic
V	= volume of the body which follows the body as it deforms
$\bar{V}$	= arbitrary vector
Δ	= first variation
ρ	= mass density
$\bar{\sigma}_{(n)}$	= stress vector acting on the differential surface area whose outer normal is $\bar{n}$
$\bar{\omega}$	= angular velocity vector of the noninertial reference frame in the inertial frame

Subscripts and superscripts

$(\_)$	= scalar quantity
$(\bar{\ })$	= vector quantity
$(\bar{\bar{\ }})$	= dyadic quantity
∇	= gradient operator
$(\ )'$	= dual definition (relative to a translating, rotating reference frame) of $(\ )$, which is defined relative to the inertial frame
$(\dot{\ })$	= material time derivative relative to the inertial frame
$(^{*})$	= material time derivative relative to the noninertial reference frame

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I. Introduction

THE equations of motion applicable to a deformable body are documented in a number of treatises[1] and textbooks.[2,3] In their fundamental form, the equations are irrefutable. But in the study of the dynamics of finite bodies, solutions (including numerical) rarely are developed which utilize the equations of motion in their complete form. Rather, approximate equations for the problem at hand are developed and then solved. Therefore, even if the solution is exact, it is an exact solution to an approximate set of equations. Also, if the exact equations are not available, one loses a good measure of errors inherent in the solution. This is the background that motivated the author to develop the equations of motion in the form presented in this study.

Hopefully, this formulation may hold advantages for applications not considered during their development. But there are a number of technical areas in which there is considerable activity currently, and which motivated this study, such as structural dynamics, aero- and hydroelasticity, finite element formulations for large deflections, and dynamics and control of flexible space vehicles. Documentation of studies related to the last of these areas is especially active, and a brief mention of the kind of analysis performed is enlightening.

Orbiting satellites have encountered problems in maintaining the precise orientation demanded by their function. Apparently, the flexibility of the structure was neglected during the design and analysis of the early spacecraft, but several of them exhibited analogous behavior, which has been attributed to the effect of structural flexibility on attitude control. A recent review of experience was presented by Modi,[4] where numerous references may be found.

The types of formulations which have been used are represented by the works of Ashley,[5] Likins,[6] and Meirovitch,[7] among others. Deformation of the body is represented by a sum of modes of deformation, starting with the "rigid body" modes. The deflections of the body relative to the "rigid body" are linearized as in treatments of the infinitesimal vibration of elastic structures. This has led to vagueness in the studies. For example, the motion of a set of axes, which in some way represents angular momentum of the body, is called "rigid body motion," "mean motion," "gross motion," "overall motion," etc. Furthermore, attempts to ascribe the motion of these axes to definitive properties of the body have not met with universal agreement[5,8] Unfortunately, the differences between the alternative interpretations are hidden within the approximations made in the formulations. In the analysis that follows, the global equations of motion are developed exactly. Body rotations are related to a rotating reference frame, which is defined rigorously in terms of the angular momentum of the body. Motion of this reference frame clearly represents what has been referred to vaguely as "rigid body motion."

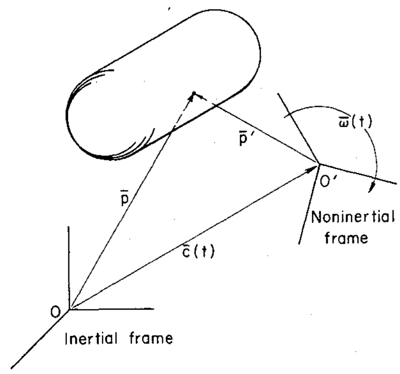

Fig. 1 Body configuration measured relative to two reference frames.

II. Classical Foundations for a Deformable Body

Euler's laws[1-3] are taken to represent the balance of momentum for a deformable body:

$$\bar{F}=\dot{\bar{P}},\ \bar{L}^{[0]}=\dot{\bar{H}}^{[0]} \tag{1}$$

In these equations, the linear and angular momenta and the time rate of change are defined relative to an inertial reference frame, and the superscript $[0]$ denotes that the torque and angular momentum are measured relative to the origin of the inertial reference frame. The applied loads and momenta for a body can be expressed in terms of the stress tensor, body forces, and velocity field of the body:

$$\bar{F}\equiv \oint_S \bar{\sigma}_{(n)}\,\mathrm{d}a+\int_V \bar{f}\,\mathrm{d}m \tag{2a}$$

$$\bar{L}^{[0]}\equiv \oint_S \bar{p}\times\bar{\sigma}_{(n)}\,\mathrm{d}a+\int_V \bar{p}\times\bar{f}\,\mathrm{d}m \tag{2b}$$

$$\bar{P}\equiv\int_V \dot{\bar{p}}\,\mathrm{d}m \tag{2c}$$

$$\bar{H}^{[0]}\equiv\int_V \bar{p}\times\dot{\bar{p}}\,\mathrm{d}m \tag{2d}$$

where V and S follow the deforming body. Substituting these expressions into Eq. (1) gives the following form of Euler's laws:

$$\oint_S \bar{\sigma}_{(n)}\,\mathrm{d}a+\int_V \bar{f}\,\mathrm{d}m=\frac{d}{dt}\int_V \dot{\bar{p}}\,\mathrm{d}m \tag{3a}$$

$$\oint_S \bar{p}\times\bar{\sigma}_{(n)}\,\mathrm{d}a+\int_V \bar{p}\times\bar{f}\,\mathrm{d}m=\frac{\mathrm{d}}{\mathrm{d}t}\int_V \bar{p}\times\dot{\bar{p}}\,\mathrm{d}m \tag{3b}$$

where $\dot{\bar{p}}$, which is the material time derivative of $\bar{p}$, is the particle velocity. Now the application of Green's theorem leads to Cauchy's laws of motion:

$$\rho\ddot{\bar{p}}=\bar{\nabla}\cdot\bar{\bar{T}}+\rho\bar{f},\quad \bar{\bar{T}}=\bar{\bar{T}}^T \tag{4}$$

where $\bar{\bar{T}}$ is the stress dyadic and $\bar{\bar{T}}^T$ is its transpose. It will be convenient to have an expression representing the conservation of kinetic energy. The desired expression is not independent of the preceding equations; rather it is a consequence of Eq. (4). To derive it, Cauchy's first law is multiplied by $\dot{\bar{p}}$, then integrated over the body, and then, using Green's transformation, one derives the following[1] equation:

$$\dot{K}=\int_V \bar{f}\cdot\dot{\bar{p}}\,\mathrm{d}m+\oint_S \bar{\sigma}_{(n)}\cdot\dot{\bar{p}}\,\mathrm{d}a-\int_V \bar{\bar{T}}:(\bar{\nabla}\dot{\bar{p}})\,\mathrm{d}v \tag{5}$$

where

$$K\equiv\int_V \frac{1}{2}\dot{\bar{p}}^2\mathrm{d}m \tag{6}$$

Since $\bar{\bar{T}}$ is symmetric, only the symmetric part of $\nabla\dot{\bar{p}}$ contributes to the last term in Eq. (5). But the symmetric part of $\nabla\dot{\bar{p}}$ is the stretching dyadic $\bar{\bar{D}}$; therefore, that term may be written as follows:

$$\int_V \bar{\bar{T}}:(\bar{\nabla}\dot{\bar{p}})\,\mathrm{d}v=\int_V \bar{\bar{T}}:\bar{\bar{D}}\,\mathrm{d}v \tag{7}$$

where

$$\bar{\bar{D}}\equiv \tfrac{1}{2}(\bar{\nabla}\,\dot{\bar{p}}\bar{\nabla}) \tag{8}$$

III. Kinematics and Indifference for a Change of Frame

The motion of the body (and all kinematic parameters) has been defined relative to an inertial reference frame. Closely associated is the time derivative (material derivative), which is also unique to the inertial frame. Sometimes, however, it is desirable to refer the configuration and motion of the body to a different reference frame, which is moving and rotating relative to the inertial frame (see Fig. 1). The relations between the position vectors (for the same material point) and material derivatives of an arbitrary vector relative to the two frames[9] are†

$$\bar{p}=\bar{c}(t)+\bar{p}',\ \dot{\bar{v}}=\bar{\omega}\times\bar{v}+\overset{*}{\bar{v}} \tag{9}$$

From these equations, the following relations can be developed:

$$\dot{\bar{p}}=\dot{\bar{c}}(t)+\dot{\bar{p}}'=\dot{\bar{c}}(t)+\bar{\omega}\times\bar{p}'+\overset{*}{\bar{p}}' \tag{10a}$$

$$\ddot{\bar{p}}=\ddot{\bar{c}}(t)+\dot{\bar{\omega}}\times\bar{p}'+\bar{\omega}\times(\bar{\omega}+\bar{p}')+2\bar{\omega}\times\overset{*}{\bar{p}}'+\overset{**}{\bar{p}}' \tag{10b}$$

Vector and tensor quantities that transform between these two reference frames similarly to tensor transformation laws are called indifferent. These quantities are identical when referred to the two reference frames, and, therefore, in the analysis that follows, the unprimed notation could have been used for both frames. The indifference is represented formally as follows:

$$\bar{\sigma}'_{(n)}=\bar{\sigma}_{(n)},\ \bar{f}'=\bar{f},\ \bar{\bar{T}}'=\bar{\bar{T}},\ \bar{\bar{D}}'=\bar{\bar{D}} \tag{11}$$

The stress vector and stress dyadic are indifferent by definition because they are defined in terms of surface forces and surface orientation, which are assumed independent of the observer's reference frame.[10] Also, the body force ($\bar{f}$) is assumed indifferent. Alternatively, the stretching dyadic indifference follows rigorously the kinematics of a continuous body.[1]

†Since this time rate of change is a material derivative (i.e., following a material point in the body), the operation transforms exactly as particle mechanics.

IV. Equations Transformed to an Arbitrary Reference Frame

It is desired to derive the governing equations, summarized by Eqs. (1) and (6), in terms of quantities relative to an arbitrary reference frame. First, relative to the reference frame, the applied loads, momenta, and kinetic energy are given the following dual definitions:

$$\bar F' \equiv \oint_S \bar\sigma'_{(n)}\, da + \int_V \bar f'\, dm \tag{12a}$$

$$\bar L'^{[0']} \equiv \oint_S \bar p' \times \bar\sigma'_{(n)}\, da + \int_V \bar p' \times \bar f'\, dm \tag{12b}$$

$$\bar P' \equiv \int_V \dot{\bar p}'\, dm \tag{12c}$$

$$\bar H'^{[0']} \equiv \int_V \bar p' \times \dot{\bar p}'\, dm \tag{12d}$$

$$K' \equiv \int_V \tfrac{1}{2}\dot{\bar p}'^{2}\, dm \tag{12e}$$

where 0′ is the origin of the reference frame.

Next $\bar p$ and $\dot{\bar p}$ are substituted from Eqs. (9) and (10) into Eqs. (2) and (6), and the terms are rearranged to derive the following relations:

$$\bar F = \bar F' \tag{13a}$$

$$\bar L^{[0]} = \bar c \times \bar F' + \bar L'^{[0']} \tag{13b}$$

$$\bar P = \dot{\bar c}M + \bar\omega \times \int_V \bar p'\, dm + \bar P' \tag{13c}$$

$$\bar H^{[0]} = \bar c \times \bar P - \dot{\bar c} \times \int_V \bar p'\, dm + \bar\omega \cdot \bar{\bar I} + \bar H'^{[0']} \tag{13d}$$

$$K = \tfrac{1}{2}\dot{\bar c}^2 M + \dot{\bar c} \cdot \left[\bar\omega \times \int_V \bar p'\, dm\right] + \dot{\bar c}\cdot\bar P' + \bar\omega\cdot\bar H'^{[0']} + \tfrac{1}{2}\bar\omega \cdot \bar{\bar I} \cdot \bar\omega + K' \tag{13e}$$

where

$$M \equiv \int_V dm \tag{14a}$$

$$\omega \equiv |\bar\omega| \tag{14b}$$

$$\bar n \cdot \bar{\bar I} \equiv \int_V \bar p' \times (\bar n \times \bar p')\, dm \tag{14c}$$

$$\bar n_\omega \equiv \bar\omega/\omega \tag{14d}$$

Terms on the right side of Eq. (5) are transformed similarly:

$$\int_V \bar f \cdot \dot{\bar p}\, dm + \oint_S \bar\sigma_{(n)} \cdot \dot{\bar p}\, da = \dot{\bar c}\cdot\bar F' + \bar\omega\cdot\bar L'^{[0']} + \int_V \bar f' \cdot \dot{\bar p}'\, dm + \oint_S \bar\sigma'_{(n)} \cdot \dot{\bar p}'\, da \tag{15}$$

Also using Eqs. (9) and (10) with (13c), (13d), (13e) the following relations are derived‡:

$$\dot{\bar P} = \ddot{\bar c}M + \dot{\bar\omega} \times \int_V \bar p'\, dm + \bar\omega \times \left[\bar\omega \times \int_V \bar p'\, dm + 2\bar P'\right] + \dot{\bar P}' \tag{16a}$$

$$\dot{\bar H}^{[0]} = \dot{\bar c} \times \bar P - \ddot{\bar c} \times \int_V \bar p'\, dm + \dot{\bar\omega}\cdot\bar{\bar I} + \bar\omega \times (\bar\omega\cdot\bar{\bar I}) + \bar\omega\cdot\dot{\bar{\bar I}} + \bar\omega\times\bar H'^{[0']} + \dot{\bar H}'^{[0']} \tag{16b}$$

$$\dot K = \ddot{\bar c} \cdot \left[\dot{\bar c}M + \bar P' + \bar\omega \times \int_V \bar p'\, dm\right] + \dot{\bar c}\cdot\left[\dot{\bar\omega}\times\int_V \bar p'\, dm + \bar\omega\times\left[\bar\omega\times\int_V \bar p'\, dm\right] + 2\bar\omega\times\bar P' + \dot{\bar P}'\right] + \dot{\bar\omega}\cdot\bar{\bar I}\cdot\bar\omega + \tfrac{1}{2}\bar\omega\cdot\dot{\bar{\bar I}}\cdot\bar\omega + \dot{\bar\omega}\cdot\bar H'^{[0']} + \bar\omega\cdot\dot{\bar H}'^{[0']} + \dot K' \tag{16c}$$

Equations (13a), (13b), (15), and (16) are substituted into (1) and (5), and the results are simplified:

$$\bar F' = \ddot{\bar c}M + \dot{\bar\omega} \times \int_V \bar p'\, dm + \bar\omega \times \left[\bar\omega \times \int_V \bar p'\, dm + 2\bar P'\right] + \dot{\bar P}' \tag{17a}$$

$$\bar L'^{[0']} = -\ddot{\bar c} \times \int_V \bar p'\, dm + \dot{\bar\omega}\cdot\bar{\bar I} + \bar\omega \times (\bar\omega\cdot\bar{\bar I}) + \bar\omega\cdot\dot{\bar{\bar I}} + \bar\omega\times\bar H'^{[0']} + \dot{\bar H}'^{[0']} \tag{17b}$$

$$\int_V \dot{\bar p}\cdot\bar f'\, dm + \oint_S \dot{\bar p}\cdot\bar\sigma'_{(n)}\, da = \ddot{\bar c}\cdot\bar P' + \dot{\bar\omega}\cdot\bar H'^{[0']} - \tfrac{1}{2}\bar\omega\cdot\dot{\bar{\bar I}}\cdot\bar\omega + \dot K' + \int_V \bar{\bar T}' : \bar{\bar D}'\, dv \tag{17c}$$

These three equations are the equations governing the motion of the body (the first two represent momentum conservation, and the third represents kinetic energy conservation) in terms of quantities measured relative to an arbitrary noninertial reference frame.

Before the equations can be applied to specific studies, the reference frame must be selected; i.e., rules must be specified for evaluating $\bar c$ and $\bar\omega$. Several different reference frames have been found to be useful. Selections are made generally in order to simplify the equations and/or their solutions for the particular problem at hand. A couple of the more common choices are considered in the next section.

V. Origin of Reference Frame Fixed to Mass Center

It is very common to associate the origin of the reference frame (0′) with the mass center of the body. This association is equivalent to the following rule for the evaluation of $\bar c$:

$$\bar c = \frac{1}{M}\int_V \bar p\, dm \tag{18}$$

The following equations are consequences of Eq. (18), which are derived easily:

$$\int_V \bar p'\, dm = \bar P' = \dot{\bar P}' = 0 \tag{19}$$

These equations indicate the obvious; if the reference frame is attached to the mass center of the body, then the body possesses zero first mass moment and linear momentum as seen by an observer in that reference frame. The governing equations reduce to the following set:

$$\bar F' = \ddot{\bar c}M \tag{20a}$$

‡The author thanks D.C. Leigh, University of Kentucky, for showing an algebraic error made in the original derivation of (16a)

$$\bar{L}'^{[0']} = \dot{\bar{\omega}} \cdot \bar{\bar{I}} + \bar{\omega} \times (\bar{\omega} \cdot \bar{\bar{I}}) + \bar{\omega} \cdot \dot{\bar{\bar{I}}} + \bar{\omega} \times \bar{H}'^{[0']} + \dot{\bar{H}}'^{[0']} \tag{20b}$$

$$\int_V \overset{*}{\bar{p}} \cdot \bar{f}' \, dm + \oint_S \overset{*}{\bar{p}} \cdot \bar{\sigma}'_{(n)} \, da = \dot{\bar{\omega}} \cdot \bar{H}'^{[0']} - \tfrac{1}{2} \bar{\omega} \cdot \dot{\bar{\bar{I}}} \cdot \bar{\omega} + \dot{K}' + \int_V \bar{\bar{T}}' : \bar{\bar{D}}' \, dv \tag{20c}$$

It is apparent that all of the complexities in the governing equations are not eliminated. Next, consider the selection of $\bar{\omega}$; two choices are discussed.

VI. Reference Frame Selections

Nonrotating Frame

A frame that has found use in the study of rigid bodies is one that does not rotate relative to the inertial frame:

$$\bar{\omega} = 0 \tag{21}$$

From this selection, the governing equations (20) reduce to the following set:

$$\bar{F}' = \ddot{\bar{c}} M \tag{22a}$$

$$\bar{L}'^{[0']} = \dot{\bar{H}}'^{[0']} \tag{22b}$$

$$\int_V \overset{*}{\bar{p}} \cdot \bar{f}' \, dm + \oint_S \overset{*}{\bar{p}} \cdot \bar{\sigma}'_{(n)} \, da = \dot{K}' + \int_V \bar{\bar{T}}' : \bar{\bar{D}}' \, dv \tag{22c}$$

The equations are greatly simplified. Actually, the simplification is only apparent for many applications, because the complexities reappear when the momentum and kinetic energy are evaluated. The next choice of frame is more enlightening for general considerations.

Rotating Frame

The equation representing the conservation of linear momentum was simplified considerably by choosing $\bar{c}$ to follow the mass center motion. By analogy, $\bar{\omega}$ will be associated with the angular momentum of the body. Similar to Eq. (18), the following rule is selected for the evaluation of $\bar{\omega}$:

$$\bar{\omega} \cdot \bar{\bar{I}} = \bar{H}^{[0]} - \bar{c} \times \bar{P} \tag{23}$$

An obvious consequence of Eq. (23) is

$$\bar{H}'^{[0']} = \overset{*}{\bar{H}}'^{[0']} = 0 \tag{24}$$

i.e., the body possesses zero angular momentum as seen by an observer in the reference frame. The expressions for the momenta and kinetic energy from (13) reduce to the following relations:

$$\bar{P} = \dot{\bar{c}} M \tag{25a}$$

$$\bar{H}^{[0]} = \bar{c} \times \bar{P} + \bar{\omega} \cdot \bar{\bar{I}} \tag{25b}$$

$$K = \tfrac{1}{2} \dot{c}^2 M + \tfrac{1}{2} \bar{\omega} \cdot \bar{\bar{I}} \cdot \bar{\omega} + K' \tag{25c}$$

Expressions for the momenta are identical to those for a rigid body, whereas the expression for the kinetic energy is modified only by the addition of K'. K' is positive definite, and its magnitude is minimized by the preceding choice of $\bar{\omega}$.

To demonstrate this property, return to Eqs. (13d) and (13e), solving for K', assume that the origin of the frame coincides with the mass center, but leave the choice of $\bar{\omega}$ arbitrary:

$$K' = \left[K - \tfrac{1}{2} \dot{c}^2 M \right] + \left[\tfrac{1}{2} \bar{\omega} \cdot \bar{\bar{I}} \cdot \bar{\omega} - \bar{\omega} \cdot (\bar{H}^{[0]} - \bar{c} \times \bar{P}) \right] \tag{26}$$

Take the variation of K' as $\bar{\omega}$ varies:

$$\Delta K' = \left[\bar{\omega} \cdot \bar{\bar{I}} - (\bar{H}^{[0]} - \bar{c} \times \bar{P}) \right] \cdot \Delta \bar{\omega} \tag{27}$$

Therefore, K' is a minimum§ when the bracketed term is zero, i.e., when $\bar{\omega}$ satisfies Eq. (23). In other words, the kinetic energy of the body as seen by an observer in the reference frame is minimized by this choice of frame.

Finally, Eqs. (20), the governing equations, reduce to the following relations:

$$\bar{F}' = \ddot{\bar{c}} M \tag{28a}$$

$$\bar{L}'^{[0']} = \dot{\bar{\omega}} \bar{\bar{I}} + \bar{\omega} \times (\bar{\omega} \cdot \dot{\bar{\bar{I}}}) + \bar{\omega} \cdot \dot{\bar{\bar{I}}} \tag{28b}$$

$$\int_V \overset{*}{\bar{p}} \cdot \bar{f}' \, dm + \oint_S \overset{*}{\bar{p}} \cdot \bar{\sigma}'_{(n)} \, da = -\tfrac{1}{2} \bar{\omega} \cdot \dot{\bar{\bar{I}}} \cdot \bar{\omega} + \dot{K}' + \int_V \bar{\bar{T}}' : \bar{\bar{D}}' \, dv \tag{28c}$$

VII. Applications

Rigid Motion

A motion is instantaneously rigid if the stretching dyadic is zero everywhere at that instant:

$$\bar{\bar{D}} = 0 \tag{29}$$

It is interesting to apply the equations to this motion, and they should compare with the classical equations for a rigid body. The motion of the body consistent with (29), (8), and (23) is given uniquely by the following:

$$\dot{\bar{p}} = \bar{\omega} \times (\bar{p} - \bar{c}) + \dot{\bar{c}} \tag{30}$$

Therefore, Eq.(30) taken together with (9) and (10) requires that the body appear stationary to an observer in the reference frame:

$$\overset{*}{\bar{p}}' = 0 \tag{31}$$

Furthermore, Eq. (12e) requires

$$K' = 0 \tag{32}$$

Next, differentiate Eq. (14c), accounting for Eq. (31):

$$\dot{\bar{n}} \cdot \bar{\bar{I}} + \bar{n} \cdot \dot{\bar{\bar{I}}} = \int_V \bar{p}' \times (\dot{\bar{n}} \times \bar{p}') \, dm \tag{33}$$

But Eq. (14c) implies

$$\dot{\bar{n}} \cdot \bar{\bar{I}} = \int_V \bar{p}' \times (\dot{\bar{n}} \times \bar{p}') \, dm \tag{34}$$

§The possibility that K' is a maximum is eliminated easily by considering a rigid body.

The preceding two equations require

$$\bar{n}\cdot\dot{\bar{\bar{I}}}=0 \tag{35}$$

But $\bar{n}$ is a unit vector of arbitrary direction; therefore.

$$\dot{\bar{\bar{I}}}=0 \tag{36}$$

Now the governing equations can be summarized using (29), (31), (32), and (36) to simplify the more general relations. First, (25) becomes

$$\bar{P}=\dot{\bar{c}}\,M \tag{37a}$$

$$\bar{H}^{[0]}=\bar{c}\times\bar{P}+\bar{\omega}\cdot\bar{\bar{I}} \tag{37b}$$

$$K=\tfrac{1}{2}\,\dot{c}^2M+\tfrac{1}{2}\,\bar{\omega}\cdot\bar{\bar{I}}\cdot\bar{\omega} \tag{37c}$$

Eqs. (28a) and (28b) become

$$\bar{F}'=\ddot{\bar{c}}\,M \tag{38a}$$

$$\bar{L}'^{[0']}=\dot{\bar{\omega}}\cdot\bar{\bar{I}}+\bar{\omega}\times(\bar{\omega}\cdot\bar{\bar{I}}) \tag{38b}$$

and each term in Eq. (28c) is identically zero. This system of equations for rigid motion is identical to the classical equations for a rigid body.

Steady Force-Free Motion

An interesting application for this theory is the question of steady motion in a force-free environment. First, the governing equations are summarized for zero forces:

$$\ddot{\bar{c}}=0 \tag{39a}$$

$$\dot{\bar{\omega}}\cdot\bar{\bar{I}}+\bar{\omega}\times(\bar{\omega}\cdot\bar{\bar{I}})+\bar{\omega}\cdot\dot{\bar{\bar{I}}}=0 \tag{39b}$$

$$-\tfrac{1}{2}\bar{\omega}\cdot\dot{\bar{\bar{I}}}\cdot\bar{\omega}+\dot{K}'+\int_V\bar{\bar{T}}':\bar{\bar{D}}'\,\mathrm{d}v=0 \tag{39c}$$

Suppose at an instant the motion is rigid. Then Eq. (39c) is identically zero, and Eq. (39b) becomes

$$\dot{\bar{\omega}}\cdot\bar{\bar{I}}+\bar{\omega}\times(\bar{\omega}\cdot\bar{\bar{I}})=0 \tag{40}$$

If, at this instant, $\bar{\omega}$=defines a principal direction of $\bar{\bar{I}}$, it follows (40) that

$$\dot{\bar{\omega}}=0 \tag{41}$$

Of course, if the motion remains rigid for all time, then Eq. (41) represents the steady motion possible for a rigid body.

A generalization of steady motion for a deforming body is possible if we seek only a steady direction for $\bar{\omega}$. A solution is obtained if $\bar{\omega}$ defines a principal direction of both $\bar{\bar{I}}$ and $\dot{\bar{\bar{I}}}$ simultaneously. For this case, Eqs. (39b) and (39c) yield the following solution:

$$\omega I_{\omega\omega}=C \tag{42a}$$

$$\dot{\omega}=-\frac{2}{C}\left[\dot{K}'+\int_V\bar{\bar{T}}':\bar{\bar{D}}'\,\mathrm{d}v\right] \tag{42b}$$

where C is a constant of integration.

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